Mathematics, Analysis

1) Let $n \geq 1$ be an integer. We endow $\mathbb{R}^n$ with the usual scalar product given by

$$\langle x, y\rangle = \sum_{k=1}^{n} x_k y_k$$

for any $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and any $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. We denote by $\|\cdot\|$ the norm defined by $\|x\|^2 = \langle x, x\rangle$. Then $\mathbf{M}_n(\mathbb{R})$, the space of $n \times n$ matrices, is equipped with the norm

$$\|T\|_2 = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|}$$

where the supremum is taken over all x in $\mathbb{R}^n$ with $x \neq 0$.

Given T in $\mathbf{M}_n(\mathbb{R})$, we denote by T^* the adjoint (or transpose) matrix of T.

a) Prove that $\|ST\|_2 \leq \|S\|_2 \|T\|_2$, $\|T^*\|_2 = \|T\|_2$ and $\|T^*T\|_2 = \|T\|_2^2$.

b) Assume for this question that $T^* = T$. Show that, for any integer N in $\mathbb{N}$,

$$\left\|T^N\right\|_2 = \|T\|_2^N .$$

c) Deduce that for any T in $\mathbf{M}_n(\mathbb{R})$ and any integer N in $\mathbb{N}$,

$$\left\|\left(T^*T\right)^N\right\|_2 = \|T\|_2^{2N} .$$

2) Consider a sequence $(T_j)_{j\in\mathbb{Z}}$ of matrices in $\mathbf{M}_n(\mathbb{R})$. Assume that there exists a function $\omega\colon \mathbb{Z} \to]0, +\infty[$ such that $\sum_{i\in\mathbb{Z}} \omega(i) < +\infty$ and such that, for any pair $(j, k) \in \mathbb{Z} \times \mathbb{Z}$,

$$\sqrt{\left\|T_j^* T_k\right\|_2} \leq \omega(j - k), \tag{1}$$

$$\sqrt{\left\|T_j T_k^*\right\|_2} \leq \omega(j - k). \tag{2}$$

The goal of this question is to prove that, for any finite subset F of $\mathbb{Z}$,

$$\left\|\sum_{j\in F} T_j\right\|_2 \leq \sigma \quad \text{where} \quad \sigma = \sum_{i\in\mathbb{Z}} \omega(i). \tag{3}$$

a) Prove that $\|T_j\|_2 \leq \omega(0)$ for any $j \in \mathbb{Z}$.

b) Fix an integer $N \geq 1$ and a finite subset F of $\mathbb{Z}$. Denote by $I = F^{2N} = \overbrace{F \times \cdots \times F}^{2N \text{ times}}$ the set of $(2N)$-uples of integers in F. Prove that for any $(i_1, i_2, \ldots, i_{2N}) \in I$,

$$\left\|T_{i_1}^* T_{i_2} T_{i_3}^* T_{i_4} \cdots T_{i_{2N-1}}^* T_{i_{2N}}\right\|_2 \leq \omega(0)\omega(i_1 - i_2)\omega(i_2 - i_3)\cdots\omega(i_{2N-2} - i_{2N-1})\omega(i_{2N-1} - i_{2N}).$$

c) Deduce that

$$\sum_{i_1}\sum_{i_2}\cdots\sum_{i_{2N-1}}\sum_{i_{2N}}\left\|T_{i_1}^*T_{i_2}\,T_{i_3}^*T_{i_4}\,\cdots T_{i_{2N-1}}^*T_{i_{2N}}\right\|_2 \le (\mathrm{card\,F})\sigma^{2\mathrm{N}}$$

where σ is as defined in (3) and the summation is taken over all $(i_1, i_2, \ldots, i_{2N-1}, i_{2N})$ in I.

d) Prove that $U = \sum_{j\in F} T_j$ satisfies the desired estimate (3).

3) Let $K\colon \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a C^1 function satisfying, for some constant $B > 0$,

i) $|K(x)| \le B|x|^{-1}$ and the derivative satisfies $|K'(x)| \le B|x|^{-2}$ for any $x \ne 0$,

ii) $K(-x) = -K(x)$ for any $x \ne 0$.

a) It is admitted that there is a C^∞ function $\varphi\colon \mathbb{R} \to \mathbb{R}$ such that

1. $\varphi(x) = \varphi(-x)$;
2. $\varphi(x) = 1$ if $|x| \le 1$ and $\varphi(x) = 0$ for $|x| \ge 2$.

For any j in $\mathbb{Z}$ define $K_j(x)$ by $K_j(x) = \big(\varphi(2^{-j}x) - \varphi(2^{1-j}x)\big)K(x)$. Prove that

$$\int_{\mathbb{R}} K_j(x)\,dx = 0, \qquad \sup_{j\in\mathbb{Z}}\sup_{x\in\mathbb{R}} 2^{2j}\left|K_j'(x)\right| < +\infty,$$

$$\sup_{j\in\mathbb{Z}}\int_{\mathbb{R}} |K_j(x)|\,dx < +\infty, \quad \sup_{j\in\mathbb{Z}} 2^{-j}\int_{\mathbb{R}} |x|\,|K_j(x)|\,dx < +\infty.$$

b) Given two integers j, k in $\mathbb{Z}$ with $j \ge k$, set

$$K_{j,k}(x) = \int_{\mathbb{R}} K_k(y)K_j(x+y)\,dy.$$

Prove that there exists a constant C_1 (independent of j, k, x) such that $|K_{j,k}(x)| \le C_1 2^{k-2j}$. Deduce that

$$\int_{\mathbb{R}} |K_{j,k}(x)|\,dx \le C_2 2^{-|j-k|}$$

for some fixed positive constant C_2 independent of j and k.

Remark. Let us admit that the estimate (3) holds when $\mathbb{C}^n$ is replaced by a Hilbert space (then T_j are bounded operators). Also, instead of finite sums, one can consider infinite sums of operators T_j that satisfy the bounds (1) and (2). The previous analysis allows to prove that the operator T defined (formally) by

$$Tu(x) = \int_{\mathbb{R}} K(x-y)u(y)\,dy$$

is bounded on L^2.